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On Oscillation of Fourth Order Delay Differential Equations

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ABSTRACT

The purpose of this paper is to obtain the oscillatory behavior of solutions of a class of fourth order differential equations

$$\left(a(t)\left(x'''(t)\right)^{\gamma}\right)' + b(t)f(x(\tau(t))) = 0 \quad for \quad t \geq t_0$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\gamma}(s)} ds < \infty$$

and also two examples are given to clarify our results.

Keywords: Oscillation and fourth order delay differential equation.

1. Introduction

In recent years the oscillation behavior of solutions of nonlinear differential equations has been studied in some papers and books. We refer the reader to the books (Agarwal et al. (2002), Agarwal et al. (2003), Agarwal et al. (2004), Agarwal et al. (2000), Kiguradze and Chanturia (1992), Ladde et al. (1987)) and the papers (Agarwal et al. (1997), Agarwal et al. (2001), Elabbasy and Moaaz (2016), Győri and Ladas (1991), Li et al. (2011b), Li and Rogovchenko (2014), Li and Rogovchenko (2017), Philos (1981), Zhang et al. (2011), Zhang et al. (2013)). In the last few years, there have been many papers which include the oscillatory theory of fourth order differential equations (Agarwal et al. (2006), Elabbasy et al. (2017), Grace et al. (2013), Grace et al. (2019), Kamo (2011), Kusano et al. (2011), Li et al. (2011a), Li et al. (2014a), Li et al. (2014b), Li and Rogovchenko (2014), Li et al. (2015), Moaaz et al. (2017), Tripathy (2013), Tripathy et al. (2013), Zhang et al. (2012), Zhang et al. (2014), Zhang et al. (2015)).

This paper deals with the oscillation of the following nonlinear fourth order delay differential equation

$$\left(a(t)\left(x'''(t)\right)^{\gamma}\right)' + b(t)f(x(\tau(t))) = 0 \quad for \quad t \geq t_0. \tag{1}$$

During this paper, we assume that the following conditions hold:

(δ_1) γ is a quotient of odd positive integers,

(δ_2) $a \in C^1[t_0, \infty)$, $a'(t) \geq 0$, $a(t) > 0$, b, $\tau \in C[t_0, \infty)$, $b(t) \geq 0$, $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$,

(δ_3) $\dfrac{f(x)}{x^{\gamma}} \geq m > 0$ for $x \neq 0$,

(δ_4) $\int_{t_0}^{\infty} \dfrac{1}{a^{1/\gamma}(s)} ds < \infty$.

There is a function $x(t) \in C^3[p_x, \infty)$, $p_x \geq t_0$ such that $a(t)(x'''(t))^{\gamma} \in C^1[p_x, \infty)$ and if it satisfies Eq.(1) on $[p_x, \infty)$ then it is a solution of Eq.(1). We restrain ourselves to the analysis of the solutions $x(t)$ of Eq.(1) with property $\sup\{| x(t) |: t \geq q\} > 0$ for all $q \geq p_x$. A solution of Eq.(1) is called oscillatory if it has arbitrarily large zeros on $[p_x, \infty)$; otherwise, it is called nonoscillatory. Eq.(1) is said to be oscillatory if all solutions of this equation are oscillatory.

For using in proofs of the theorems, we need to give the following lemma.

Lemma 1.1. *(Agarwal et al. (2000), Lemma 2.2.3) Let $y \in C^n([t_0,\infty)(0,\infty))$. Assume that $y^n(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$ and that there exists $t_1 \geq t_0$ such that $y^{(n-1)}(t)y^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} y(t) \neq 0$, then, for every $\lambda \in (0,1)$, there exists $t_k \in [t_1,\infty)$ such that*

$$y(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} |y^{(n-1)}(t)| \quad for \quad t \in [t_k,\infty).$$

2. Main Results

In this section, we obtain oscillation criteria for the solutions of Eq.(1). For convenience, we introduce the symbols

$$A(t) = \int_t^\infty \frac{1}{a^{1/\gamma}(s)} ds, \quad \phi'_+(t) = \max\{0,\phi'(t)\}, \quad \psi'_+(t) = \max\{0,\psi'(t)\}.$$

Lemma 2.1. *If $x(t)$ is an eventually positive three times continuously differentiable function such that $a(t)(x'''(t))^\gamma$ is continuously differentiable and $(a(t)(x'''(t))^\gamma)' \leq 0$ for large t, then one of the following cases holds for large t.*

(θ_1) $x'(t) > 0,\ x''(t) > 0,\ x'''(t) > 0,\ x^{(iv)}(t) \leq 0.$

(θ_2) $x'(t) > 0,\ x''(t) < 0,\ x'''(t) > 0,\ x^{(iv)}(t) \leq 0.$

(θ_3) $x'(t) < 0,\ x''(t) > 0,\ x'''(t) < 0.$

(θ_4) $x'(t) > 0,\ x''(t) > 0,\ x'''(t) < 0.$

The proof is clear.

Theorem 2.1. *Let $(\delta_1),(\delta_2),(\delta_3)$ and (δ_4) hold. Suppose that there exist positive functions $\phi,\psi \in C^1(t_0,\infty)$ such that*

$$\int_{t_0}^\infty \left[mb(s)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma \phi(s) - \frac{2^\gamma}{(\gamma+1)^{\gamma+1}} \frac{a(s)(\phi'_+(s))^{\gamma+1}}{(k_1\phi(s)s^2)^\gamma} \right] ds = \infty, \qquad (2)$$

$$\int_{t_0}^{\infty}\left[\psi(s)\int_s^{\infty}\left[\frac{m}{a(v)}\int_v^{\infty}b(\zeta)\left(\frac{\tau(\zeta)}{\zeta}\right)^{\gamma}d\zeta\right]^{\frac{1}{\gamma}}dv-\frac{(\psi'_{+}(s))^2}{4\psi(s)}\right]ds=\infty, \quad (3)$$

$$\int_{t_0}^{\infty}\left[mb(s)\left(\int_s^{\infty}\int_u^{\infty}A(v)dvdu\right)^{\gamma}-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{\int_s^{\infty}A(v)dv}{\int_s^{\infty}\int_u^{\infty}A(v)dvdu}\right]ds=\infty \quad (4)$$

and

$$\int_{t_0}^{\infty}\left[mb(s)\left(\frac{k_2}{2}\tau^2(s)\right)^{\gamma}A^{\gamma}(s)-\frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\frac{1}{A(s)a^{1/\gamma}(s)}\right]ds=\infty \quad (5)$$

for some constants $k_1, k_2 \in (0,1)$, *then every solution of Eq.(1) is oscillatory.*

Proof. Let x be a nonoscillatory solution of Eq.(1) on the interval $[t_0,\infty)$. Without loss of generality, we can assume that x is eventually positive. From Lemma 2.1, there exists a $t_1 \geq t_0$ such that $x(t)$ has one of four cases $(\theta_1)-(\theta_4)$ for $t \geq t_1$. For (θ_1), by Kiguradze Lemma (Kiguradze and Chanturia (1992)), we find $x(t) \geq \dfrac{t}{3}x'(t)$, hence

$$\frac{x(\tau(t))}{x(t)} \geq \frac{\tau^3(t)}{t^3}. \quad (6)$$

We define

$$z(t)=\phi(t)\frac{a(t)(x''')^{\gamma}(t)}{x^{\gamma}(t)}, \quad t \geq t_1. \quad (7)$$

Then $z(t) > 0$ and

$$z'(t)=\phi'(t)\frac{a(t)(x''')^{\gamma}(t)}{x^{\gamma}(t)}+\phi(t)\frac{(a(x''')^{\gamma})'(t)}{x^{\gamma}(t)}-\gamma\phi(t)\frac{x^{\gamma-1}(t)x'(t)a(t)(x''')^{\gamma}(t)}{x^{2\gamma}(t)}. \quad (8)$$

From Lemma 1.1, we get

$$x'(t) \geq \frac{k_1}{2}t^2x'''(t) \quad (9)$$

for every $k_1 \in (0,1)$ and all sufficiently large t. Hence, by (8) and (9), we find

$$z'(t) \leq \phi'(t)\frac{a(t)(x''')^{\gamma}(t)}{x^{\gamma}(t)}+\phi(t)\frac{(a(x''')^{\gamma})'(t)}{x^{\gamma}(t)}-\frac{\gamma k_1}{2}t^2\phi(t)\frac{x'''(t)a(t)(x''')^{\gamma}(t)}{x^{\gamma+1}(t)}. \quad (10)$$

Thus, because of (1), (6) and (δ_3), we obtain

$$z'(t) \leq \frac{\phi'_+(t)}{\phi(t)} z(t) - m\phi(t)b(t)\left(\frac{\tau^3(t)}{t^3}\right)^\gamma - \frac{\gamma k_1}{2}\frac{t^2}{(a(t)\phi(t))^{1/\gamma}} z^{(\gamma+1)/\gamma}(t). \quad (11)$$

With

$$\alpha = \frac{\gamma k_1 t^2}{2(a(t)\phi(t))^{1/\gamma}}, \quad \beta = \frac{\phi'_+(t)}{\phi(t)}, \quad y = z(t)$$

by using the inequality

$$\beta y - \alpha y^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{\beta^{\gamma+1}}{\alpha^\gamma}, \quad \alpha, \beta > 0, \quad (12)$$

and from (11), we get

$$z'(t) \leq -mb(t)\left(\frac{\tau^3(t)}{t^3}\right)^\gamma \phi(t) + \frac{2^\gamma}{(\gamma+1)^{\gamma+1}}\frac{a(t)(\phi'_+(t))^{\gamma+1}}{(k_1\phi(t)t^2)^\gamma},$$

which implies that

$$\int_{t_1}^{t}\left[mb(s)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma \phi(s) - \frac{2^\gamma}{(\gamma+1)^{\gamma+1}}\frac{a(s)(\phi'_+(s))^{\gamma+1}}{(k_1\phi(s)s^2)^\gamma}\right] ds \leq z(t_1)$$

for every $k_1 \in (0,1)$ and all sufficiently large t. Therefore this contradicts (2).

Presume that (θ_2) is satisfied. By integrating (1) from t to l, we have

$$a(l)(x''')^\gamma(l) - a(t)(x''')^\gamma(t) + \int_t^l b(s)f(x(\tau(s)))ds = 0.$$

From Kiguradze Lemma, we get $\dfrac{x(\tau(t))}{x(t)} \geq \dfrac{\tau(t)}{t}$. By using (δ_3) and $x'(0) > 0$,

$$a(l)(x''')^\gamma(l) - a(t)(x''')^\gamma(t) + x^\gamma(t)\int_t^l mb(s)\left(\frac{\tau(s)}{s}\right)^\gamma ds \leq 0.$$

Letting $l \longrightarrow \infty$, we find the inequality

$$-a(t)(x''')^\gamma(t) + x^\gamma(t)\int_t^\infty mb(s)\left(\frac{\tau(s)}{s}\right)^\gamma ds \leq 0,$$

and so,

$$-x'''(t) + x(t)\left[\frac{m}{a(t)}\int_t^\infty b(s)\left(\frac{\tau(s)}{s}\right)^\gamma ds\right]^{1/\gamma} \leq 0.$$

Taking the integral again from t to ∞, we get

$$x''(t) + x(t) \int_t^\infty \left[\frac{m}{a(v)} \int_v^\infty b(s) \left(\frac{\tau(s)}{s} \right)^\gamma ds \right]^{1/\gamma} dv \leq 0. \tag{13}$$

Now, we define

$$\xi(t) = \psi(t) \frac{x'(t)}{x(t)}, \quad t \geq t_1.$$

Then $\xi(t) > 0$ for $t \geq t_1$ and differentiating the above equality and using (13), we obtain

$$\xi'(t) \leq -\psi(t) \int_t^\infty \left[\frac{m}{a(v)} \int_v^\infty b(s) \left(\frac{\tau(s)}{s} \right)^\gamma ds \right]^{1/\gamma} dv + \frac{\psi'_+(t)}{\psi(t)} \xi(t) - \frac{\xi^2(t)}{\psi(t)}. \tag{14}$$

Thus, we have

$$\xi'(t) \leq -\psi(t) \int_t^\infty \left[\frac{m}{a(v)} \int_v^\infty b(s) \left(\frac{\tau(s)}{s} \right)^\gamma ds \right]^{1/\gamma} dv + \frac{(\psi'_+(t))^2}{4\psi(t)}.$$

Hence, we get

$$\int_{t_1}^t \left[\psi(s) \int_s^\infty \left[\frac{m}{a(v)} \int_v^\infty q(\zeta) \left(\frac{\tau(\zeta)}{\zeta} \right)^\gamma d\zeta \right]^{\frac{1}{\gamma}} dv - \frac{(\psi'_+(s))^2}{4\psi(s)} \right] ds \leq \xi(t_1),$$

which contradicts (3).

Consider case (θ_3) holds. Since $a(x''')^\gamma$ is nonincreasing, we have

$$a^{1/\gamma}(s)x'''(s) \leq a^{1/\gamma}(t)x'''(t), \quad s \geq t \geq t_1.$$

Integrating this inequality from t to l, we get

$$x''(l) \leq x''(t) + a^{1/\gamma}(t)x'''(t) \int_t^l a^{-1/\gamma}(s) ds.$$

Letting $l \longrightarrow \infty$, we see that

$$x''(t) \geq -a^{1/\gamma}(t)x'''(t)A(t). \tag{15}$$

Taking the integral (15) from t to ∞, we obtain

$$-x'(t) \geq \int_t^\infty -a^{1/\gamma}(s)x'''(s)A(s)ds \geq -a^{1/\gamma}(t)x'''(t) \int_t^\infty A(s)ds. \tag{16}$$

By taking the integral (16) from t to ∞, we find

$$x(t) \geq -\int_t^\infty a^{1/\gamma}(u)x'''(u)\int_u^\infty A(s)dsdu \geq -a^{1/\gamma}(t)x'''(t)\int_t^\infty\int_u^\infty A(s)dsdu. \tag{17}$$

Next, we define

$$\nu(t) = \frac{a(t)(x''')^\gamma(t)}{x^\gamma(t)}, \quad t \geq t_1. \tag{18}$$

Thus, $\nu(t) < 0$ for $t \geq t_1$, by (1), (16) and (δ_3), we conclude that

$$\nu'(t) \leq -mb(t)\frac{x^\gamma(\tau(t))}{x^\gamma(t)} - \gamma\frac{a^{(\gamma+1)/\gamma}(t)(x''')^{\gamma+1}(t)}{x^{\gamma+1}(t)}\int_t^\infty A(s)ds. \tag{19}$$

Hence, by (18) and (19), we obtain

$$\nu'(t) \leq -mb(t) - \gamma\nu^{(\gamma+1)/\gamma}(t)\int_t^\infty A(s)ds. \tag{20}$$

From (17), we have

$$\nu(t)\left(\int_t^\infty\int_u^\infty A(s)dsdu\right)^\gamma \geq -1. \tag{21}$$

Multiplying (20) by $\left(\int_t^\infty\int_u^\infty A(s)dsdu\right)^\gamma$ and integrating from t_1 to t, we find

$$\begin{aligned}
&\left(\int_t^\infty\int_u^\infty A(s)dsdu\right)^\gamma \nu(t) - \left(\int_{t_1}^\infty\int_u^\infty A(s)dsdu\right)^\gamma \nu(t_1) \\
&\quad +\gamma\int_{t_1}^t\int_s^\infty A(v)dv\left(\int_s^\infty\int_u^\infty A(v)dvdu\right)^{\gamma-1}\nu(s)ds \\
&\quad + \int_{t_1}^t mb(s)\left(\int_s^\infty\int_u^\infty A(v)dvdu\right)^\gamma ds \\
&+\gamma\int_{t_1}^t \nu^{(\gamma+1)/\gamma}(s)\left(\int_s^\infty\int_u^\infty A(v)dvdu\right)^\gamma\int_s^\infty A(v)dvds \leq 0.
\end{aligned}$$

We set

$$\begin{aligned}
\beta &= \int_s^\infty A(v)dv\left(\int_s^\infty\int_u^\infty A(v)dvdu\right)^{\gamma-1}, \\
\alpha &= \left(\int_s^\infty\int_u^\infty A(v)dvdu\right)^\gamma\int_s^\infty A(v)dv, \\
y &= -\nu(s).
\end{aligned}$$

Using the inequality

$$-\beta y+\alpha y^{\frac{\gamma+1}{\gamma}}\geq-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{\beta^{\gamma+1}}{A^{\gamma}},\quad \alpha,\beta>0, \tag{22}$$

we conclude that

$$\begin{aligned}&\int_s^\infty A(v)dv\left(\int_s^\infty\int_u^\infty A(v)dvdu\right)^{\gamma-1}\nu(s)\\&+\nu^{(\gamma+1)/\gamma}(s)\left(\int_s^\infty\int_u^\infty A(v)dvdu\right)^{\gamma}\int_s^\infty A(v)dv\\&\qquad\geq-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{\int_s^\infty A(v)dv}{\int_s^\infty\int_u^\infty A(v)dvdu}.\end{aligned}$$

Hence, by (21), it follows that

$$\begin{aligned}\int_{t_1}^t&\left[mb(s)\left(\int_s^\infty\int_u^\infty A(v)dvdu\right)^{\gamma}-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{\int_s^\infty A(v)dv}{\int_s^\infty\int_u^\infty A(v)dvdu}\right]ds\\&\qquad\leq\left(\int_{t_1}^\infty\int_u^\infty A(s)dsdu\right)^{\gamma}\nu(t_1)+1\end{aligned}$$

which contradicts (4).

Assume that (θ_4) holds. Because of the proof of case (θ_3), we know (15). From Lemma 1.1, we get

$$x(t)\geq\frac{k_2}{2}t^2x''(t) \tag{23}$$

for every $k_2\in(0,1)$ and all sufficiently large t. We define

$$\mu(t)=\frac{a(t)(x''')^{\gamma}(t)}{(x'')^{\gamma}(t)},\quad t\geq t_1. \tag{24}$$

Then $\mu(t)<0$ for $t\geq t_1$ and, by virtue of (23), (24) and (δ_3), we conclude that

$$\mu'(t)\leq-mb(t)\left(\frac{k_2}{2}\tau^2(t)\right)^{\gamma}-\gamma\frac{\mu^{(\gamma+1)/\gamma}(t)}{a^{1/\gamma}(t)}. \tag{25}$$

Multiplying this inequality by $A^{\gamma}(t)$ and integrating from t_1 to t, we find

$$\begin{aligned}A^{\gamma}(t)\mu(t)-A^{\gamma}(t_1)\mu(t_1)+\gamma\int_{t_1}^t a^{-1/\gamma}(s)A^{\gamma-1}(s)\mu(s)ds&\\\leq-\int_{t_1}^t mb(s)\left(\frac{k_2}{2}\tau^2(s)\right)^{\gamma}A^{\gamma}(s)ds&\\-\gamma\int_{t_1}^t\frac{\mu^{(\gamma+1)/\gamma}(s)}{a^{1/\gamma}(s)}A^{\gamma}(s)ds.&\end{aligned}$$

By applying the same steps in case (θ_3), we obtain

$$\left[\int_{t_1}^{t} mb(s)\left(\frac{k_2}{2}\tau^2(s)\right)^{\gamma} A^{\gamma}(s) - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\frac{1}{A(s)a^{1/\gamma}(s)}\right] ds \le A^{\gamma}(t_1)\mu(t_1)+1.$$

Therefore this contradicts (5) and this completes the proof. $\square$

Remark 2.1. *Taking $f(x(\tau(t))) = x^{\gamma}(\tau(t))$ in Eq.(1), Theorem 2.1 in Zhang et al. (2014) is obtained.*

In the following theorem, we compare the oscillatory behavior of Eq.(1) with the second order differential equations which are given in Agarwal et al. (1997) and Swanson (1968).

Theorem 2.2. *Suppose that $(\delta_1), (\delta_2), (\delta_3)$ and (δ_4) are satisfied. If the differential equations*

$$\left(\frac{a(t)}{t^{2\gamma}}(x'(t))^{\gamma}\right)' + mb(t)\left(\frac{k_1\tau^3(t)}{2t^3}\right)^{\gamma} x^{\gamma}(t) = 0, \tag{26}$$

$$x''(t) + x(t)\int_t^{\infty}\left[\frac{m}{a(v)}\int_v^{\infty} q(s)\left(\frac{\tau(s)}{s}\right)^{\gamma} ds\right]^{1/\gamma} dv = 0, \tag{27}$$

$$\left(\left(\int_t^{\infty} A(s)ds\right)^{-\gamma}(x'(t))^{\gamma}\right)' + mb(t)x^{\gamma}(t) = 0 \tag{28}$$

and

$$(a(t)(x'(t))^{\gamma})' + mb(t)\left(\frac{k_2}{2}\tau^2(t)\right)^{\gamma} x^{\gamma}(t) = 0 \tag{29}$$

are oscillatory for some constants $k_1, k_2 \in (0,1)$. Then every solution of Eq.(1) is oscillatory.

Proof. As proof of Theorem 2.1, we have that (11), (14), (20) and (25). If we set $\phi(t) = 1$ in (11), then we get

$$z'(t) + \frac{\gamma k_1}{2}\frac{t^2}{(a(t))^{1/\gamma}} z^{(\gamma+1)/\gamma} + mb(t)\left(\frac{\tau^3(t)}{t^3}\right)^{\gamma} \le 0$$

for every constant $k_1 \in (0,1)$. Thus, from Agarwal et al. (1997), we can see that the equation (26) is nonoscillatory for every constant $k_1 \in (0,1)$. This is contradiction. Taking $\psi(t) = 1$ in (14), we get that the equation (27) is nonoscillatory from Therorem 2.15 in Swanson (1968). The other cases are similarly demonstrated. $\square$

Remark 2.2. *Taking $f(x(\tau(t))) = x^{\gamma}(\tau(t))$ in Eq.(1), Theorem 2.2 in Zhang et al. (2014) is obtained.*

3. Examples

Example 3.1. *As a special case of Eq.(1), we consider*

$$\left(t^7\left(x'''(t)\right)\right)' + \eta t^3 e^{x(t)} = 0 \quad for \quad t \geq 1, \tag{30}$$

where $\eta > 0$ is a constant. If we choose $\phi(t) = \psi(t) = 1$ and $m = 1$, then we find that (2) and (3) hold, (4) and (5) are satisfied for $\eta > 120$. Therefore, by Theorem 2.1, every solution of Eq.(30) is oscillatory for $\eta > 120$.

Example 3.2. *Consider the delay differential equation*

$$\left(e^t\left(x'''(t)\right)\right)' + e^t(x^3(t-1) + x(t-1)) = 0 \quad for \quad t \geq 0. \tag{31}$$

It is easy to see that every solution of Eq.(31) is oscillatory because of Theorem 2.1 for $m = \phi(t) = \psi(t) = 1$.

4. Conclusion

We have introduced a nonlinear fourth order delay differential equation (1). Eq.(1) is a generalization of the equation studied in Zhang et al. (2014). If it is $f(x(\tau(t))) = x^\gamma(\tau(t))$ in Eq.(1), the equation (1.1) in Zhang et al. (2014) is gotten. Moreover, we have given two theorems on oscillation and when $f(x(\tau(t))) = x^\gamma(\tau(t))$, Theorem 2.1 and Theorem 2.2 about oscillation criteria for the solutions of Eq.(1) are as Theorem 2.1 and Theorem 2.2 in Zhang et al. (2014). Finally, we have present two examples to illustrate our main results.

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